Motion Planning for LTL Specifications: A Satisfiability Modulo Convex Optimization Approach

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UC Berkeley, UCLA, and UPenn

Joint work with

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George Pappas (UPenn), and Paulo Tabuada (UCLA)

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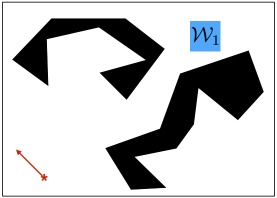

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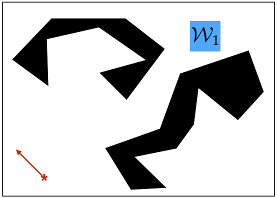

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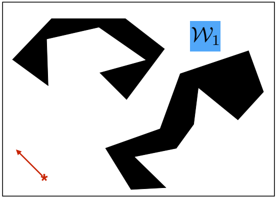

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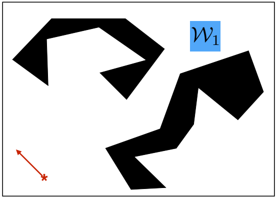

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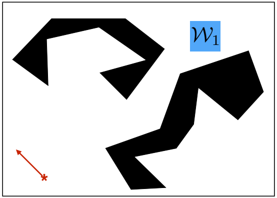

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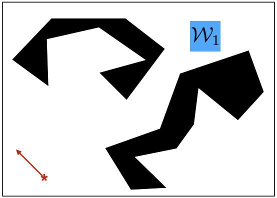

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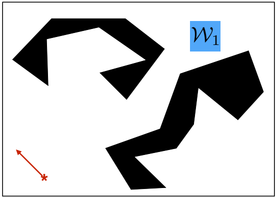
Objective:

- Generate the input sequence $u_0, u_1, \ldots, u_L$ such that the trajectory of the robot satisfies Φ.



Solver Architecture [0/3]

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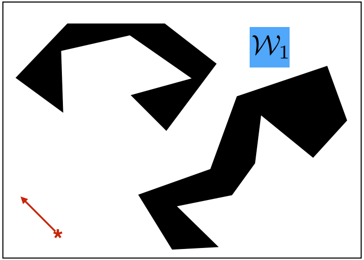

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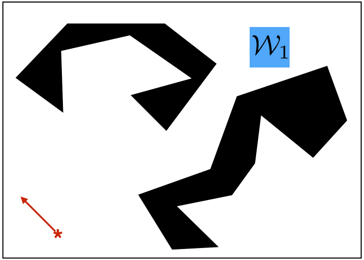

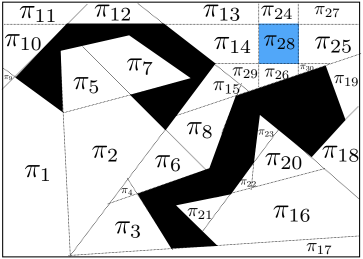

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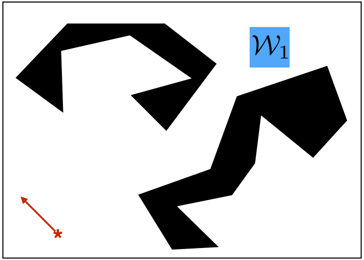

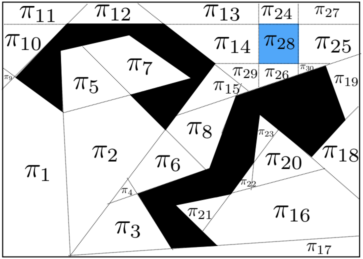

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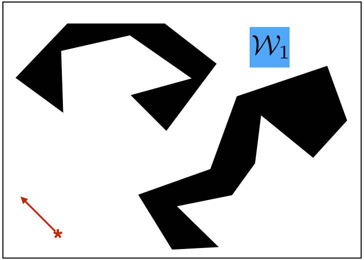

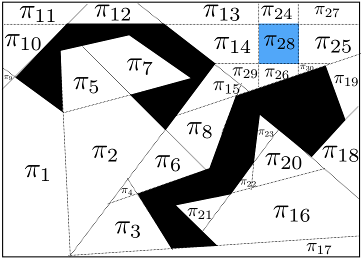

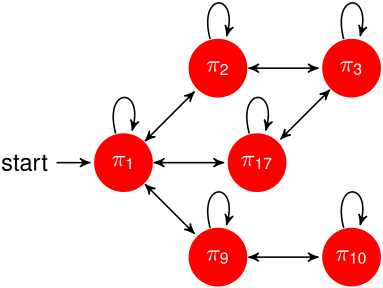

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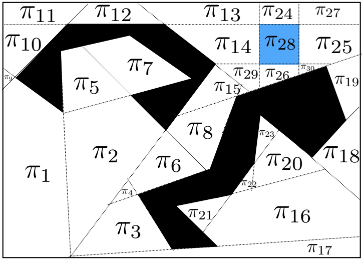

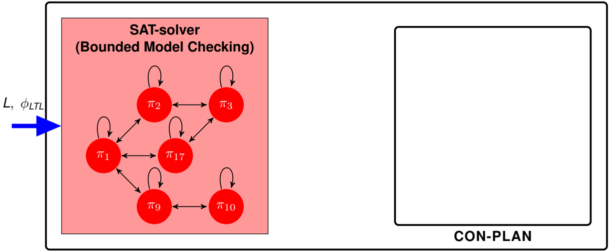

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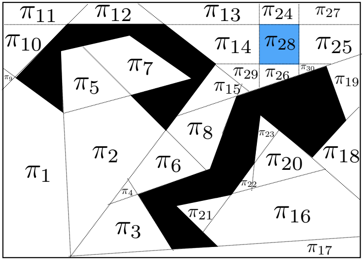

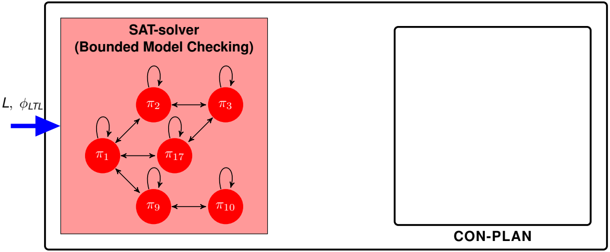

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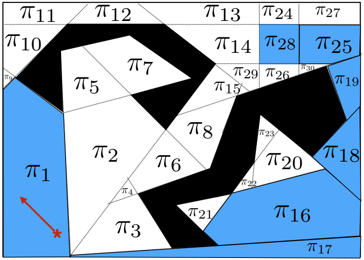

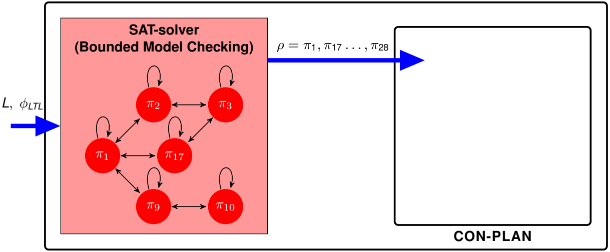

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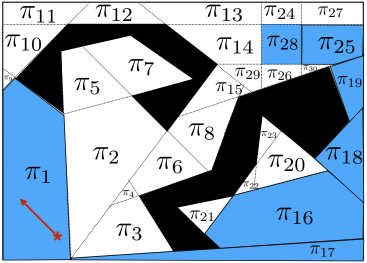

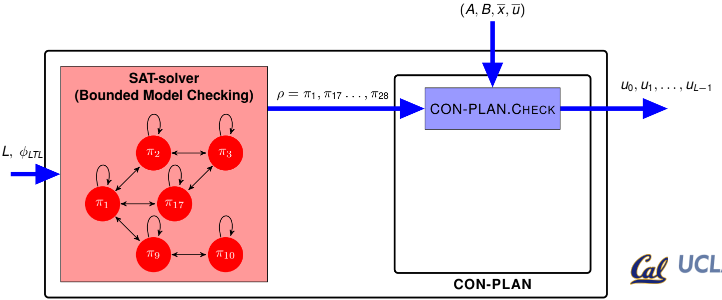

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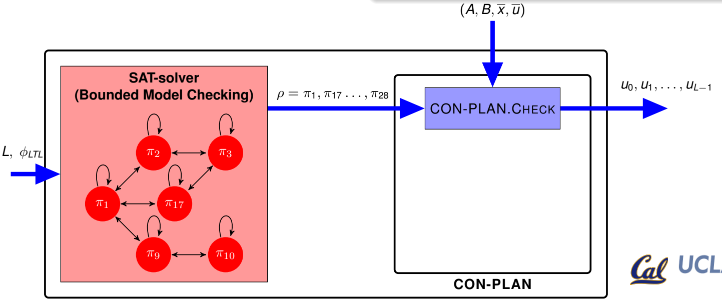

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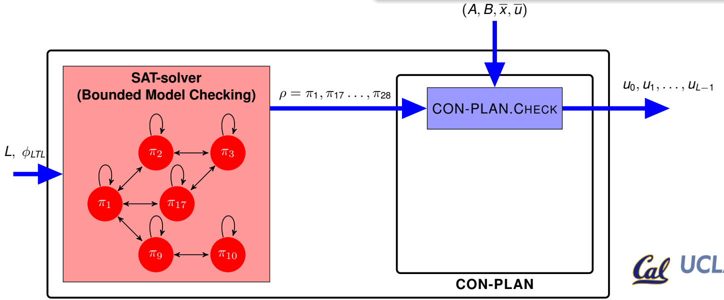

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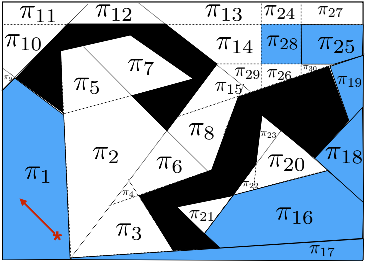

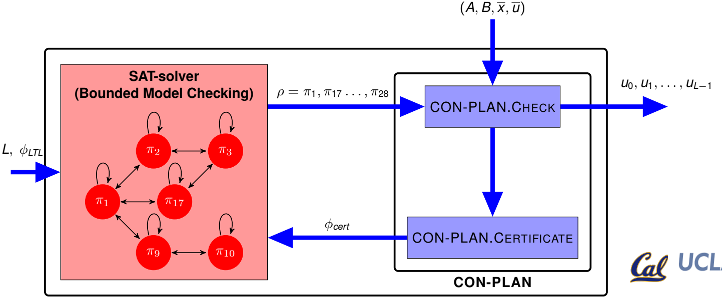

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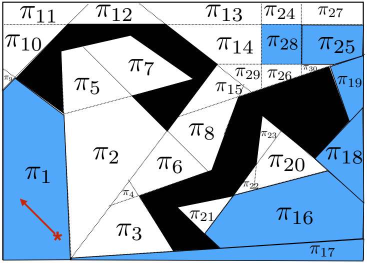

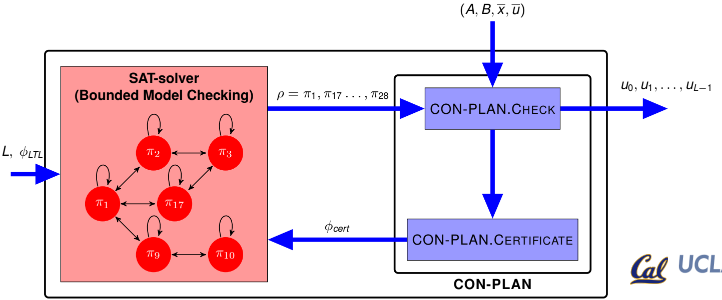

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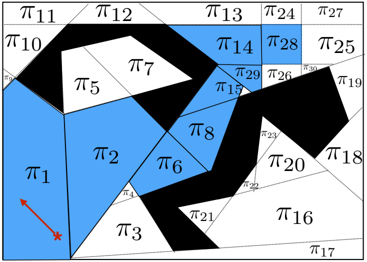

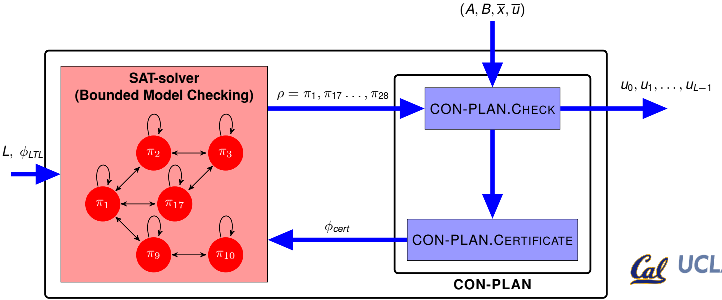

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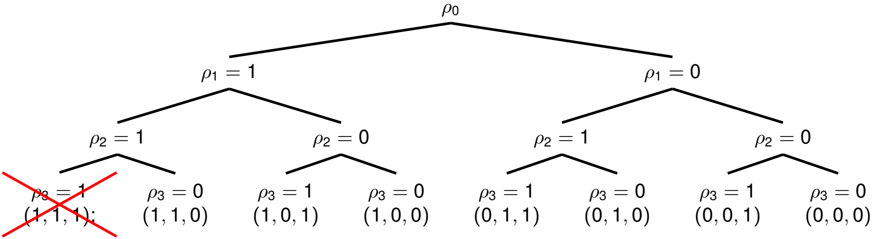

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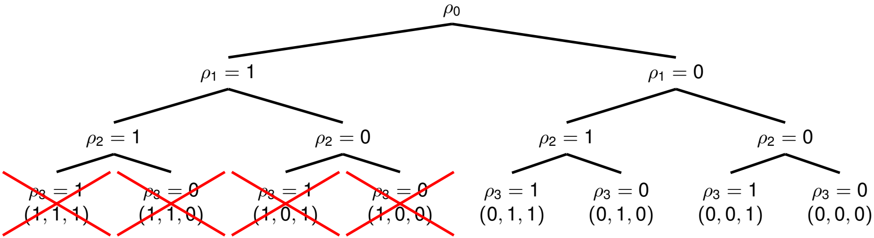

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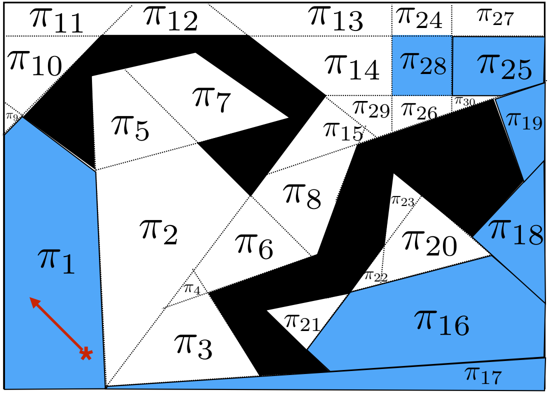
- Leads to solving multiple optimization problems.



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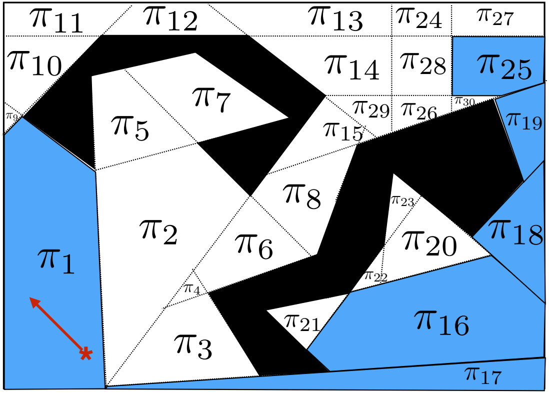

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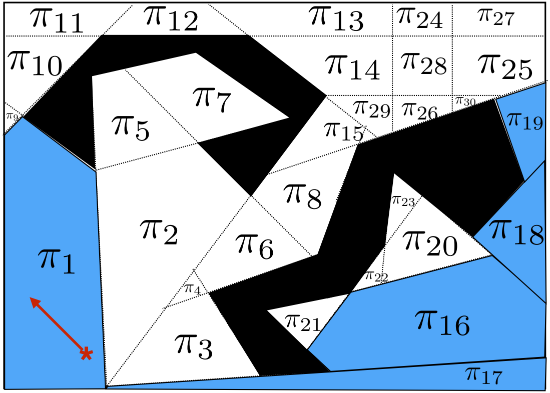

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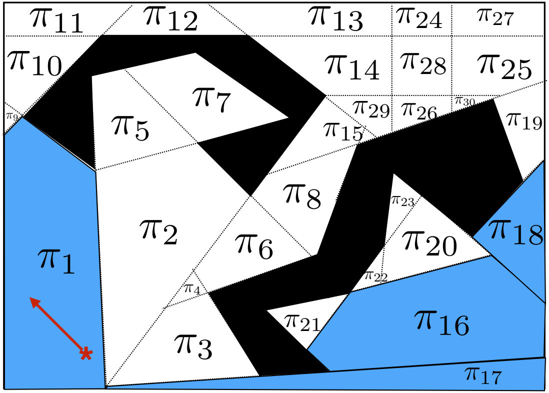

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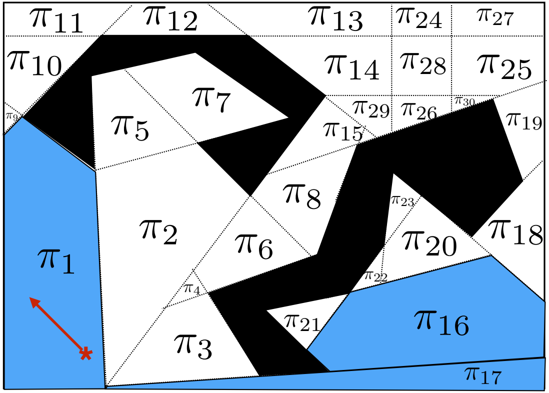

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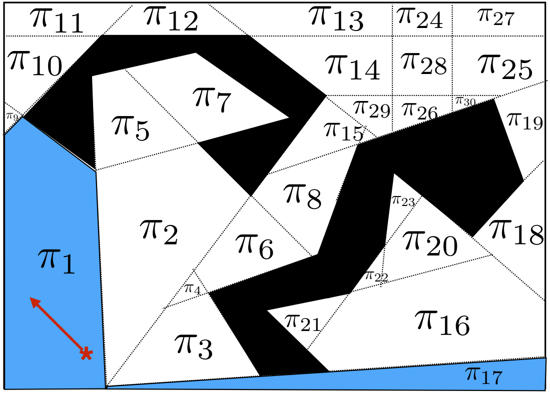

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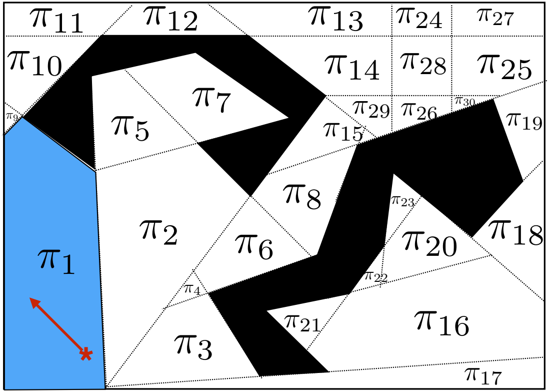

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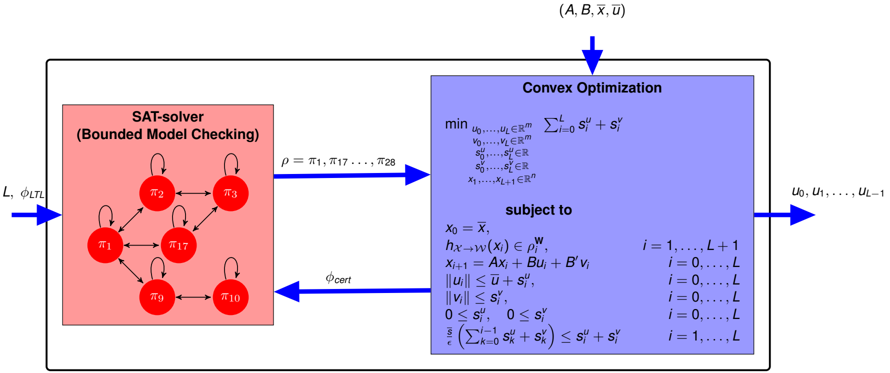

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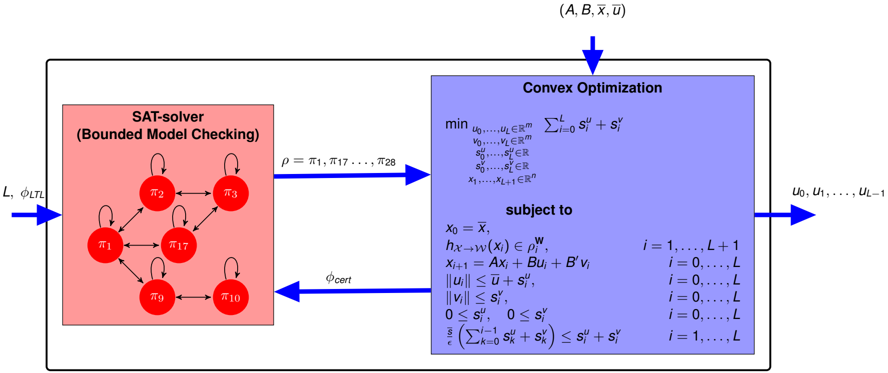

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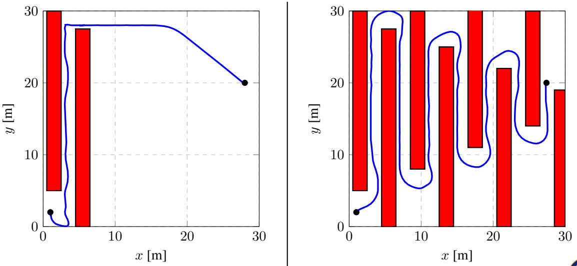

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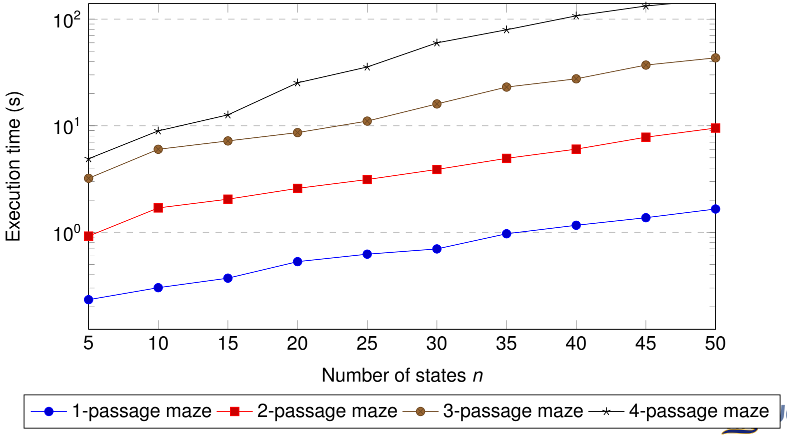
- **Workspace:** 30m $\times$ 30m maze-like workspace.
  - We increase number of passages from 1 to 4.
- We compare execution-time against (1) standard RRT and (2) LTL OPT tool (mixed-integer linear program).

Number of passages	**SMT-Based Motion Planner [s]**			**RRT [s]**	**LTL OPT [s]**
	Discrete abstraction	DIS-PLAN	CON-PLAN		
1	1.9975	0.1360	0.2542	10.4381	>7200
2	7.1461	1.1290	0.9294	122.3017	time out
3	19.3267	3.6495	1.0053	423.6957	time out
4	43.0985	4.0913	1.9204	1002.4193	time out

Case Study 2: Scalability Results

- Maze-like workspace with increasing number of passages.
- We increase the number of states n (randomly generate the matrices A and B).
- Take average across 10 runs.



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  - Security: secure state estimation.

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  - We are currently developing a comprehensive theory of Satisfiability Modulo Convex Optimization.