A Complexity Measure on Büchi Automata

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**ω-Automata**

- **ω-Automata**: automata that process infinite words
  - used for verification and synthesis of reactive systems

- There are many ways to define acceptance condition for ω-automata:
  - Simplest
    - Büchi
  - Most General
    - Muller
    - Parity
    - Rabin
    - co-Büchi
    - Streett

- We use acronyms for automata
  - **DMA** = det. Muller Automaton
  - **NBA** = non-det. Büchi Automaton

  and the class of languages they recognize
  - **DM, NB, ...**
**ω-Automata** (not that well behaved)

- Theory of ω-Automata is more involved than automata on finite words
- Variations in expressive power

\[
\begin{array}{cccccc}
\text{NB} & \text{DM} & \text{DR} & \text{DS} & \text{DP} \\
\text{NM} & \text{NR} & \text{NS} & \text{NP} \\
\end{array}
\]

\[
\begin{array}{c}
\text{DB} \\
\text{DC} \\
\end{array}
\]

\[
\begin{array}{c}
\text{NC} \\
\end{array}
\]

DB ≠ NB

DC = NC
**ω-Automata (not that well behaved)**

- Theory of ω-Automata is more involved than automata on finite words
- Variations in expressive power
- No unique minimal automaton
- No Myhill-Nerode theorem
- ...


Hierarchy of regular \( \omega \)-languages [Wagner 75]

A complexity measure for \( \omega \)-languages defined on det. Muller aut. (DM)

DM

[Wagner '75]

DR

[Krishnan et al. '95]

US

[DPP]

NB

[Perrin & Pin '97]

[This work]
Outline

- Muller
  - Aut. Def.
  - Compl. Measure. Def.

- Büchi
  - Aut. Def.
  - Compl. Measure. Def.

Chain Measure — Width Measure
The acceptance condition is a set of sets of states $\mathcal{F} = \{ F_1, F_2, \ldots, F_k \}$

A run is accepting if the set $S$ of states visited infinitely often during the run is in $\mathcal{F}$

$\mathcal{F}_1 = \{ \{q_0\}, \{q_1\} \}$

$L_1 = \{a,b\}^* (a^\omega + b^\omega)$
The Chain Measure [Wagner]

- An SCC in the automaton graph of a DMA is either accepting (in \( F \)) or rejecting (not in \( F \)).

- A chain is a sequence of sets of reachable SCCs with alternating polarities where one subsumes the other.

- It is a positive chain (resp. negative chain) if the bottom SCC is accepting (resp. rejecting).

\[ F_1 = \{ \{q_0\}, \{q_1\} \} \]

Positive \( \{q_0\} \subseteq \{q_0, q_1\} \)
The Wagner Hierarchy (chain measure)

- We use $\text{pcm}(D)$ or positive-chain measure of $D$ for the maximal positive chain in $D$.
- We use $\text{ncm}(D)$ or negative-chain measure of $D$ for the maximal negative chain in $D$.
- The classes of the Wagner hierarchy are

$$\text{DM}_k^- = \{ L \mid \exists \text{DMA} \ D : L = [D] \ & \ ncm(D) \leq k \}$$

$$\text{DM}_k^+ = \{ L \mid \exists \text{DMA} \ D : L = [D] \ & \ pcm(D) \leq k \}$$

$$\text{DM}_k^{++} = \{ L \mid \exists \text{DMA} \ D : L = [D] \ & \ ncm(D) \leq k \mid pcm(D) \leq k \}$$
The Wagner Hierarchy

Theorem

- Duality:
  \[ L \in DM_k^- \text{ iff } L^c \in DM_k^+ \]

- Strictness:
  \[ DM_k^- \subsetneq DM_{k+1}^- \text{ and } DM_k^+ \subsetneq DM_{k+1}^+ \]

- Relation to Büchi and coBüchi:
  \[ L \in DM_1^- \text{ iff } L \in DC \text{ and } L \in DM_1^+ \text{ iff } L \in DB \]

- Relation to Parity
  \[ L \in DM_k^+ \text{ iff } L \in DP_k \]
Outline

- Muller
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- Büchi
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Chain Measure

Width Measure
Büchi Automata

- Looks like an NFA, but definition of acceptance is different.
- An infinite word is accepted if it visits one of the accepting states infinitely often.
There may be many runs of an NBW on a given word \( w \).

- **Width** \((N,w)\), loosely speaking, is the maximal number of infinite runs with different suffixes.
- Formal definition is based on Kähler and Wilke’s trees.
A Complexity Measure for NBA

- Let $\text{width}(N, w)$ denote the width of NBA $N$ w.r.t. $\omega$-word $w$
- Let $\text{width}(N) = \max \{ \text{width}(N, w) \mid w \in \Sigma^\omega \}$
- The classes of the NBA hierarchy are

$\text{NB}_k = \{ L \mid \exists \text{ NBA } N : L = [N] \& \text{width}(N) \leq k \}$

**Theorem**

$L \in \text{DM}_k^+$ implies $L \in \text{NB}_{\Gamma_{k/2} - 1}$

$L \in \text{NB}_k$ implies $L \in \text{DM}_k^{2k+1}$
Outline

Muller
- Aut. Def.
- Compl. Measure. Def.

Büchi
- Aut. Def.
- Compl. Measure. Def.

Chain Measure

Width Measure
From $NB_k$ to $DM^+_k$,

$L \in NB^+_k$ implies $L \in DM^+_{2k+1}$

- Follows from “A Modular Approach to Büchi Determinization” [F. & Lustig ’15]
- It basically shows that
  
  NBA with width $k$ $\Rightarrow$ DPA with $2k+1$ colors

- By the hierarchy Theorem
  
  NBA with width $k$ $\Rightarrow$ $L \in DM^+_{2k+1}$
Outline

**Muller**
- Aut. Def.
- Compl. Measure. Def.

**Büchi**
- Aut. Def.
- Compl. Measure. Def.

Chain Measure  Width Measure
$L \in DM_1^+ \implies L \in NB_1$

- $L \in DM_1^+ \implies L \in DB$ thus $L \in NB_1$

- Direct Proof: $L \in DM_1^+$ implies $L \in DB$ thus $L \in NB_1$

We need to verify that we visit an accepting set and not just a subset of it.

We can do this using LAR!
A method to track which states are visited infinitely often
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\[ \begin{align*}
\varepsilon & : q_1 \to q_2 \to q_3 \to q_4 \to q_5 \to \# \\
a & : q_1 \to \# \to q_3 \to q_4 \to q_5 \to q_2 \\
c & : q_1 \to q_3 \to q_4 \to q_5 \to \# \to q_2 \\
b & : q_1 \to q_3 \to \# \to q_5 \to q_2 \to q_4 \\
c & : q_1 \to q_3 \to \# \to q_2 \to q_4 \to q_5
\end{align*} \]
A method to track which states are visited infinitely often

Lemma:

An SCC $S$ is visited infinitely often \textbf{iff} 
- from some point on to the right of $\#$ are only $S$ states and 
- infinitely often to the right of $\#$ are all states of $S$
$L \in \text{DM}^+_1$ implies $L \in \text{NB}_1$

- $L \in \text{DM}^+_1$ implies $L \in \text{DB}$ thus $L \in \text{NB}_1$

- **Direct Proof:**

Using $L$ we can build a $\text{DBA}$ whose accepting states are all states of the form $u \# v$ where $\text{set}(v)$ is accepting

Maximal width = 1!
\( L \in \mathcal{DM}_1 \) implies \( L \in \mathcal{NB}_2 \)

- \( L \in \mathcal{DM}_1 \) implies
\(L \in \text{DM}_1^{-1} \text{ implies } L \in \text{NB}_2\)

Given DMA partitioned to SCCs, where green components are the maximal accepting SCCs.

We build the following NBA:

Recognizes the same language

Maximal width = 2!
$L \in \text{DM}^+_k$ implies $L \in \text{NB} \left\lceil k/2 \right\rceil + 1$

- $L \in \text{DM}^+_k$ implies
$L \in \text{DM}_k^+$ implies $L \in \text{NB}_{\Gamma_{k/2}^{1/2} + 1}$

We build an NBA by making copies of any accepting SCC which is maximal within a subsuming rejecting SCC.
\[ L \in \text{EDM}_k^+ \text{ implies } L \in \text{NB}_\Gamma^{\lceil k/2 \rceil + 1} \]

We build an \textbf{NBA} by making copies of any accepting \textbf{SCC} which is \textbf{maximal} within a \textbf{subsuming rejecting SCC}.

For each copy we make a \textbf{LAR} automaton, and the accepting sets are all \( u \# v \) where \( \text{set}(v) \) is accepting.
**L \in DM^+_k** implies **L \in NB \Gamma^{k/2+1}**

The resulting NBA accepts exactly the same language.

The width of this NBA is bounded by the maximal number of "maximal subsuming accepting SCCs" + 1.

The subtle part of the argument shows that although one can non-deterministically jump to the same/subsuming SCC in various point in the run, all the runs that jump to a certain SCC eventually conjoin in the same state.
To conclude

- We have defined a complexity measure on NBAs
- We showed
  - Given an NBA we can compute its width in $n^{O(n)}$ [F. & Lustig'15]
  - We can thus bound its level on the Wagner hierarchy
  - But a lower bound for the question what is the minimal level given an NBA remains open.

Theorem

\[
L \in DM^+_k \quad \text{implies} \quad L \in NB_{\Gamma_k/2^7 + 1} \\
L \in NB_k \quad \text{implies} \quad L \in DM^+_{2k + 1}
\]
Thank you for your attention!

Questions/Comments?

Thanks to Javier Esparza for his question during the presentation of [F. & Lustig] at CONCUR'15 that prompted this work!