Synthesizing Robust Systems

Paulo Tabuada

Ayca Balkan, Sina Caliskan, Yasser Shoukry, Rupak Majumdar (MPI)

Cyber-Physical Systems Laboratory
Department of Electrical Engineering
University of California at Los Angeles

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Robustness

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- The most basic controller designs do not explicitly address robustness, but they are robust against unmodeled disturbances.
- Can the same be done for software?

Robustness

What is known about software robustness?

In Computer Science:

- Recent work by Bloem, Chatterjee, Chaudhuri, Gulwani, Henzinger, Jobstman, Majumdar, ...
- Older work by Dijkstra (self-stabilizing algorithms).

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In Control Theory:

- There is a subfield of control theory called robust control;
- The following classification will be useful:
  - State based methods (modern view) (first part of the talk);
  - Input-output based methods (older view originated from the analysis of amplifiers and other electrical circuits) (second part of the talk).

We start with a plain automaton.

Definition

A finite-state automaton is a triple $A = (Q, \Sigma, \delta)$ consisting of:

- A finite set of states Q;
- A finite set of (control) inputs Σ;
- A transition function $\delta : Q \times \Sigma \rightarrow Q$.

State based robustness

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How to reason about modest deviations from the nominal behavior?

State based robustness

Towards a definition

We introduce metric automata.

Definition

A finite-state metric automaton is a sextuple $A_\beta = (Q, d, \Sigma, X, \beta, \delta)$ consisting of:

- A finite set of states Q;
- A metric $d : Q \times Q \to \mathbb{R}_0^+$;
- A finite set of (control) inputs Σ;
- A finite set of (disturbance) inputs X including a special symbol ϵ denoting nominal (no disturbance) behavior;
- A parameter $\beta \in \mathbb{R}_0^+$ defining the "power" of the disturbance;
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- A transition function $\delta : Q \times \Sigma \times X \to Q$.

It seems that we are explicitly modeling the disturbances through the transition function δ.

State based robustness

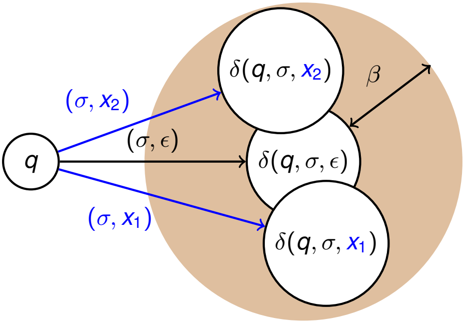
Disturbance model

Nominal transition:

$$q \xrightarrow{(\sigma, \epsilon)} \delta(q, \sigma, \epsilon)$$

State based robustness
Disturbance model

All the disturbed transitions:



$$d(\delta(q, \sigma, \epsilon), \delta(q, \sigma, x)) \leq \beta \quad \forall q \in Q, \ \sigma \in \Sigma, \ x \in X.$$

State based robustness

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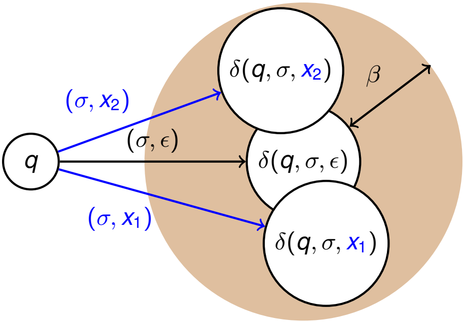

$$d(\delta(q, \sigma, \epsilon), \delta(q, \sigma, x)) \leq \beta \quad \forall q \in Q,\ \sigma \in \Sigma,\ x \in X.$$

The parameter β does not need to be known: results will be parameterized by β.

State based robustness

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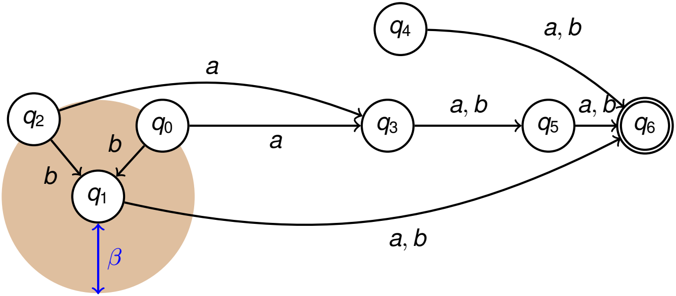
Let $F = \{q_6\}$ be a reachability objective.



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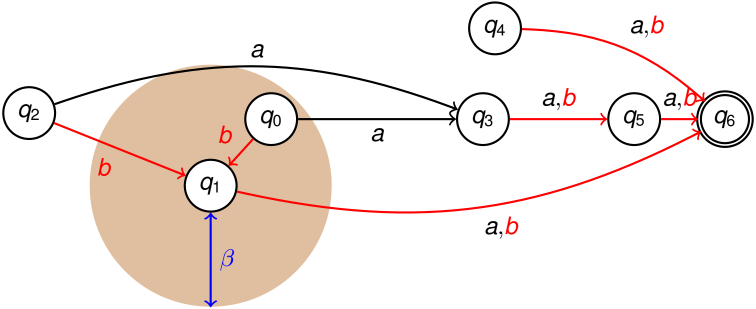
The shortest path strategy chooses the control input b at every state.



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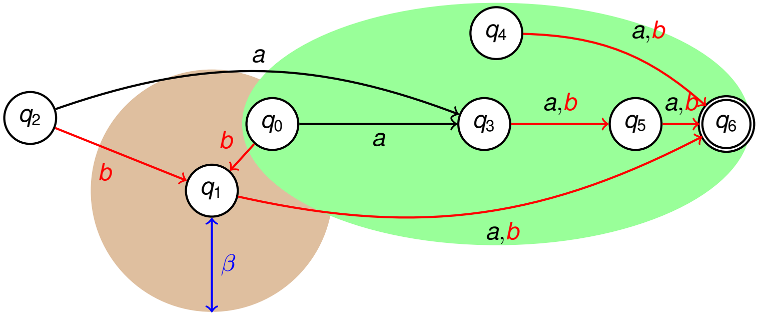

Guarantee from q_0: some state in the green ellipsis will be reached in finite time.

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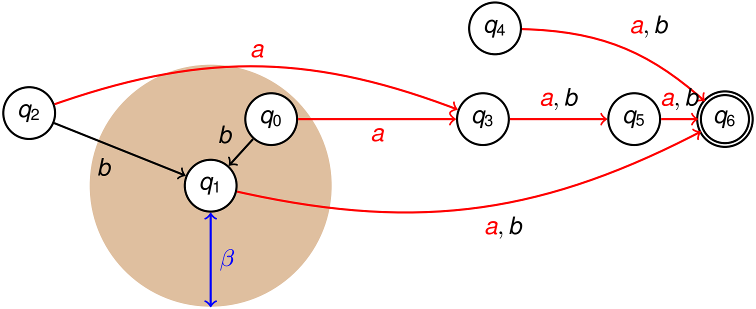

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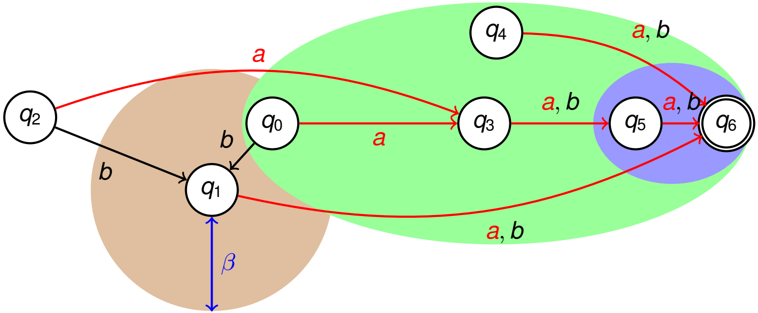

Guarantee from q_0: some state in the blue ellipsis is reached in finite time.

Some standard definitions:

- A trace $s \in Q^* \cup Q^\omega$ of the automaton A_β is a (finite or infinite) sequence of states $s = q_0 q_1 q_2 \ldots$ from Q for which there exist control inputs $\sigma_0, \sigma_1, \sigma_2, \ldots$ and disturbance inputs $x_0, x_1, x_2, \ldots$ satisfying $\delta(q_i, \sigma_i, x_i) = q_{i+1}$ for $i \geq 0$;

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- A memoryless (control) strategy for an automaton A_β is a function $S : Q \to \Sigma$ specifying a control input choice for each state $q \in Q$;
- A memoryless (control) strategy is winning for an automaton A_β if every trace of A_β complying with $S : Q \to \Sigma$ satisfies the acceptance condition.

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State based robustness

A definiton

Definition

A winning strategy for the automaton A_0 and reachability objective $F \subseteq Q$ is γ-robust if for any $\beta \in \mathbb{R}_0^+$ it is winning for the automaton A_β with reachability objective $\mathcal{B}_{\gamma\beta}(F)$:

$$\mathcal{B}_{\gamma\beta}(F) = \{q \in Q \mid d(q, F) \leq \gamma\beta\}\,.$$

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- The parameter γ describes how much F is inflated to obtain $\mathcal{B}_{\gamma\beta}(F)$.
- The map transforming environment strategies to the language accepted by A_β is uniformly continuous with modulus of continuity γ.

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State based robustness

Verification and synthesis

Given an automaton A_0, $\gamma \in \mathbb{R}_0^+$, and a strategy S one can ask:

- Verification: Is S γ-robust?
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All the above problems can be reduced to dynamic programming and are thus polynomially solvable.

All these results extend to Büchi and parity objectives[1].

[1] **A theory of ω-regular robust software synthesis**
Rupak Majumdar, Elaine Render, and Paulo Tabuada
To appear in ACM Transactions on Embedded Computing Systems.

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State based robustness requires a metric.

- What if I have two different automata defining the same language?
- How to reason about robustness before having an implementation with states?
- How to handle refinement and abstraction?

Input/output based robustness

Towards a definition

- Rather than automata we now consider transducers $f : \Sigma^* \rightarrow \Lambda^*$;

Input/output based robustness
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  - The effect of a sporadic disturbance should disappear in finitely many steps;
  - Well known requirements in control theory that recently appeared as two separate notions of robustness: [2] and [3].

---

[2] **Synthesizing Robust Systems**
R. P. Bloem, K. Greimel, T. Henzinger, B. Jobstmann
Proceedings of the 9th International Conference on Formal Methods in Computer-Aided Design, FMCAD 2009

[3] **Robustness of Sequential Circuits**
L. Doyen, T.A. Henzinger, A. Legay, and D. Nickovic
Proceedings of the 10th International Conference on Application of Concurrency to System Design, ACSD 2010.

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Based on the control theoretic notion of Input-to-State Dynamic Stability we propose:

Definition

Given parameters $\gamma, \eta \in \mathbb{N}$, we say the transducer $f : \Sigma^* \to \Lambda^*$ is (γ, η)-*Input-Output Stable (IOS)* if for each $\sigma \in \Sigma^*$ we have:

$$O(f(\sigma)) \leq \max_{\sigma' \preceq \sigma} \left\{ \gamma\, I(\sigma') - \eta\left(|\sigma| - |\sigma'|\right) \right\}.$$

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Input/output based robustness

Some test cases

Some intuition for this inequality.

$$O\left(f\left(\sigma\right)\right) \leq \max_{\sigma' \preceq \sigma} \left\{\gamma\, I\left(\sigma'\right) - \eta\left(\left|\sigma\right| - \left|\sigma'\right|\right)\right\}$$

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Consider the following sequence of input and output costs:

	σ_1	$\sigma_1\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3\sigma_4$
I	0	0	0	0
$O \circ f$	0	1	0	0

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Hence, $\max_{\sigma' \preceq \sigma} \left\{\gamma\, I\left(\sigma'\right) - \eta\left(|\sigma| - |\sigma'|\right)\right\} = \max\{0, -\eta, -2\eta\} = 0$.

IOS requires $O(f(\sigma_1\sigma_2)) = 1 \leq 0$ which does not hold!

Input/output based robustness
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Some intuition for this inequality.

$$O(f(\sigma)) \leq \max_{\sigma' \preceq \sigma} \{\gamma I(\sigma') - \eta(|\sigma| - |\sigma'|)\}$$

Consider the following sequence of input and output costs (persistent disturbance):

	σ_1	$\sigma_1\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3\sigma_4$
I	0	2	2	2
$O \circ f$	0	0	4	4

We have 3 prefixes of $\sigma = \sigma_1\sigma_2$:

Some intuition for this inequality.

$$O(f(\sigma)) \leq \max_{\sigma' \preceq \sigma} \left\{ \gamma \, I(\sigma') - \eta \left(|\sigma| - |\sigma'| \right) \right\}$$

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for $\sigma' = \sigma_1\sigma_2$ we have $\gamma I(\sigma') - \eta(|\sigma| - |\sigma'|) = \gamma 2 - \eta(2 - 2) = 2\gamma$.

Some intuition for this inequality.

$$O\left(f\left(\sigma\right)\right) \leq \max_{\sigma' \preceq \sigma} \left\{\gamma\, I\left(\sigma'\right) - \eta\left(|\sigma| - |\sigma'|\right)\right\}$$

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We have 3 prefixes of $\sigma = \sigma_1\sigma_2$:

for $\sigma' = \sigma_1$ we have $\gamma I(\sigma') - \eta(|\sigma| - |\sigma'|) = \gamma 0 - \eta(2-1) = -\eta$.

Input/output based robustness

Some test cases

Some intuition for this inequality.

$$O\left(f\left(\sigma\right)\right) \leq \max_{\sigma' \preceq \sigma} \left\{\gamma\, I\left(\sigma'\right) - \eta\left(|\sigma| - |\sigma'|\right)\right\}$$

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IOS requires $O(f(\sigma_1\sigma_2)) = 0 \leq 2\gamma$.

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IOS requires $O(f(\sigma_1\sigma_2)) = 0 \leq 2\gamma$. At this point we can take $\gamma = 0$.

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Input/output based robustness

Some test cases

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$$O(f(\sigma)) \leq \max_{\sigma' \preceq \sigma} \{\gamma I(\sigma') - \eta(|\sigma| - |\sigma'|)\}$$

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Input/output based robustness

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for $\sigma' = \sigma_1\sigma_2$ we have $\gamma I(\sigma') - \eta(|\sigma| - |\sigma'|) = \gamma 2 - \eta(3 - 2) = 2\gamma - \eta$.

Input/output based robustness
Some test cases

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We have 4 prefixes of $\sigma = \sigma_1\sigma_2\sigma_3$:

$$\text{for } \sigma' = \sigma_1 \text{ we have } \gamma I(\sigma') - \eta(|\sigma| - |\sigma'|) = \gamma 0 - \eta(3 - 1) = -2\eta.$$

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We have 4 prefixes of $\sigma = \sigma_1\sigma_2\sigma_3$:

for $\sigma' = \varepsilon$ we have $\gamma I(\sigma') - \eta(|\sigma| - |\sigma'|) = \gamma 0 - \eta(3 - 0) = -3\eta$.

Input/output based robustness

Some test cases

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$$O\left(f\left(\sigma\right)\right) \leq \max_{\sigma' \preceq \sigma} \left\{\gamma\, I\left(\sigma'\right) - \eta\left(|\sigma| - |\sigma'|\right)\right\}$$

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Hence, $\max_{\sigma' \preceq \sigma} \left\{\gamma\, I\left(\sigma'\right) - \eta\left(|\sigma| - |\sigma'|\right)\right\} = \max\{2\gamma, 2\gamma - \eta, -\eta, -2\eta\} = 2\gamma$.

Input/output based robustness

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Some intuition for this inequality.

$$O(f(\sigma)) \leq \max_{\sigma' \preceq \sigma} \{\gamma\, I(\sigma') - \eta\, (|\sigma| - |\sigma'|)\}$$

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IOS requires $O(f(\sigma_1\sigma_2\sigma_3)) = 4 \leq 2\gamma$.

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Input/output based robustness

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We have 4 prefixes of $\sigma = \sigma_1\sigma_2\sigma_3$:

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Hence, $\max_{\sigma' \preceq \sigma} \{\gamma\, I(\sigma') - \eta\,(|\sigma| - |\sigma'|)\} = \max\{2\gamma, 2\gamma - \eta, -\eta, -2\eta\} = 2\gamma$.

IOS requires $O(f(\sigma_1\sigma_2\sigma_3)) = 4 \leq 2\gamma$. A similar analysis for the remaining strings leads to IOS with $\gamma = 2$.

Some intuition for this inequality.

$$O(f(\sigma)) \leq \max_{\sigma' \preceq \sigma} \{\gamma I(\sigma') - \eta(|\sigma| - |\sigma'|)\}$$

Consider the following sequence of input and output costs (sporadic disturbance):

	σ_1	$\sigma_1\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3\sigma_4$
I	2	0	0	0
$O \circ f$	0	4	3	2

Some intuition for this inequality.

$$O\left(f\left(\sigma\right)\right) \leq \max_{\sigma' \preceq \sigma} \left\{\gamma\, I\left(\sigma'\right) - \eta\left(|\sigma| - |\sigma'|\right)\right\}$$

Consider the following sequence of input and output costs (sporadic disturbance):

	σ_1	$\sigma_1\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3\sigma_4$
I	2	0	0	0
$O \circ f$	0	4	3	2

A similar analysis leads to the following constraints:

$$\begin{aligned} O(f(\sigma_1)) &= 0 \leq 2\gamma \\ O(f(\sigma_1\sigma_2)) &= 4 \leq 2\gamma - \eta \\ O(f(\sigma_1\sigma_2\sigma_3)) &= 3 \leq 2\gamma - 2\eta \\ O(f(\sigma_1\sigma_2\sigma_3\sigma_4)) &= 2 \leq 2\gamma - 3\eta \end{aligned}$$

Some intuition for this inequality.

$$O\left(f\left(\sigma\right)\right) \leq \max_{\sigma' \preceq \sigma} \left\{\gamma\, I\left(\sigma'\right) - \eta\left(|\sigma| - |\sigma'|\right)\right\}$$

Consider the following sequence of input and output costs (sporadic disturbance):

	σ_1	$\sigma_1\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3\sigma_4$
I	2	0	0	0
$O \circ f$	0	4	3	2

A similar analysis leads to the following constraints:

$$\begin{aligned}
O(f(\sigma_1)) &= 0 \leq 2\gamma = 6 \\
O(f(\sigma_1\sigma_2)) &= 4 \leq 2\gamma - \eta = 6 - 1 = 5 \\
O(f(\sigma_1\sigma_2\sigma_3)) &= 3 \leq 2\gamma - 2\eta = 6 - 2 = 4 \\
O(f(\sigma_1\sigma_2\sigma_3\sigma_4)) &= 2 \leq 2\gamma - 3\eta = 6 - 3 = 3
\end{aligned}$$

IOS holds for $\gamma = 3$ and $\eta = 1$.

Input/output based robustness

A definition

Based on the control theoretic notion of Input-to-State Dynamic Stability we propose:

Definition

Given parameters $\gamma, \eta \in \mathbb{N}$, we say the transducer $f : \Sigma^* \to \Lambda^*$ is (γ, η)-*input-output stable* if for each $\sigma \in \Sigma^*$ we have

$$O(f(\sigma)) \leq \max_{\sigma' \preceq \sigma} \left\{ \gamma I(\sigma') - \eta \left(|\sigma| - |\sigma'| \right) \right\}.$$

- The parameter γ is called the *robustness gain*. It measures how much the disturbance is amplified.
- The parameter η is called the *rate of decay*. It measures how quickly the effects of a disturbance disappear.
- The notion of (γ, η)-input-output stability captures the two desired properties:
  - Bounded disturbances should lead to bounded consequences;
  - The effect of a sporadic disturbance should disappear in finitely many steps;

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When is a transducer IOS?

Problem ((γ, η)-IOS Verification)

Given a transducer f $: \Sigma^* \to \Lambda^*$*, input and output cost functions I* $: \Sigma^* \to \mathbb{N}_0$ *and O* $: \Lambda^* \to \mathbb{N}_0$*, respectively, and parameters* $\gamma, \eta \in \mathbb{N}$*, is the transducer f* (γ, η)*-IOS with respect to* (I, O)*?*

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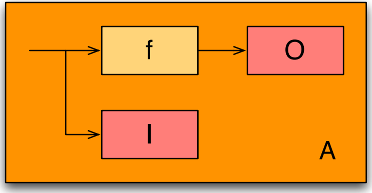
Given a transducer $f : \Sigma^* \to \Lambda^*$, input and output cost functions $I : \Sigma^* \to \mathbb{N}_0$ and $O : \Lambda^* \to \mathbb{N}_0$, respectively, and parameters $\gamma, \eta \in \mathbb{N}$, is the transducer f (γ, η)-IOS with respect to (I, O)?

Problem (IOS Verification)

Given a transducer $f : \Sigma^* \to \Lambda^*$ and input and output cost functions $I : \Sigma^* \to \mathbb{N}_0$ and $O : \Lambda^* \to \mathbb{N}_0$, respectively, does there exist $\gamma, \eta \in \mathbb{N}$ such that f is (γ, η)-IOS with respect to (I, O)? If so, find all such γ and η.

Assume that f, I, and O are defined by finite-state (weighted) automata and compose them in the single automaton A:



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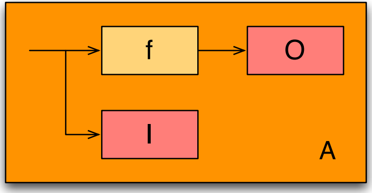

We now consider the lattice M^Q of functions from the set of states Q of A to $M = \{1, 2, \ldots, \gamma w\}$ where w is the largest weight in the automaton defining I.

Robustness

Solving the verification problem

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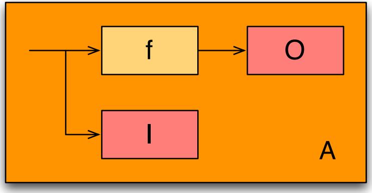

We now consider the lattice M^Q of functions from the set of states Q of A to $M = \{1, 2, \ldots, \gamma w\}$ where w is the largest weight in the automaton defining I.

On M^Q we can define the operator $F : M^Q \rightarrow M^Q$ given by:

$$F(W)(q) = \max \left\{ \gamma H^I(q), W(q), \min_{q' \in \mathrm{Pre}(q)} W(q') - \eta \right\}.$$

Theorem ((γ, η)-IOS Verification)

Let $f : \Sigma^* \to \Lambda^*$, $I : \Sigma^* \to \mathbb{N}_0$, and $O : \Lambda^* \to \mathbb{N}_0$ be defined by (weighted) finite state automata. Given $\eta, \gamma \in \mathbb{N}$, the transducer f is (γ, η)-IOS with respect to (I, O) iff the infimal fixed point of F, denoted by W^*, satisfies the following inequality for every $q \in Q$:

$$H^O(q) \leq W^*(q).$$

Note that W^* is computed in $O(|Q| \cdot |\gamma w|)$ steps.

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- For the IOS verification problem, there exists a different operator whose fixed point characterizes the existence of (γ, η) for which f is (γ, η)-IOS.
- Furthermore, we can compute all the values of γ (but only some of the values of η) for which f is (γ, η) -IOS.

How about synthesis?

How about synthesis?

- The set of inputs Σ is split as $\Sigma = \Sigma_c \times \Sigma_d$ with Σ_c being control inputs and Σ_d being disturbance inputs.

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- A controller is a map $C : \Sigma^* \times \Sigma_c \rightarrow \Sigma_c$ transforming the history of past inputs $\sigma \in \Sigma^*$ and a given control input request $\sigma^c \in \Sigma_c$ into the control input $C(\sigma, \sigma_c)$.

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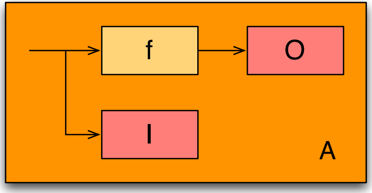
Recall the automaton A:



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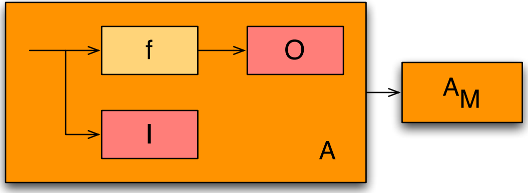
From A we can construct a monitor A_M for the (γ, η)-IOS property:



where the set of states of A_M is $M = \{1, 2, \ldots, \gamma w\}$ with w being the maximum weight of the automaton defining I.

Robustness

Solving the synthesis problem

Theorem

Let $f : \Sigma^* \to \Lambda^*$, $I : \Sigma^* \to \mathbb{N}_0$, and $O : \Lambda^* \to \mathbb{N}_0$ be defined by (weighted) finite state automata. Given $\eta, \gamma \in \mathbb{N}$, the transducer f is (γ, η)-IOS with respect to (I, O) iff every reachable state (q, m) of $A \times A_M$ satisfies $H^O(q) \leq m$.

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Theorem

Let $f : \Sigma^* \to \Lambda^*$, $I : \Sigma^* \to \mathbb{N}_0$, and $O : \Lambda^* \to \mathbb{N}_0$ be defined by (weighted) finite state automata. Given $\eta, \gamma \in \mathbb{N}$, the transducer f is (γ, η)-IOS with respect to (I, O) iff every reachable state (q, m) of $A \times A_M$ satisfies $H^O(q) \leq m$.

- This result provides a different strategy for the verification problem: verify that the set $S = \{(q, m) \in Q \times M | H^O(q) \leq m\}$ is invariant;

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- It also provides a solution to the synthesis problem: synthesize a controller to render the set S invariant;

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- This result provides a different strategy for the verification problem: verify that the set $S = \{(q, m) \in Q \times M | H^O(q) \leq m\}$ is invariant;
- It also provides a solution to the synthesis problem: synthesize a controller to render the set S invariant;
- Since safety games can be solved in linear time, the complexity of synthesizing a controller enforcing (γ, η)-IOS is linear in the size of $A \times A_M$, *i.e.*, it takes $O(|Q| \cdot |\gamma w| \cdot |\Sigma_c|)$ time.

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Robustness

Several issues remain open:

- the characterization of all the (γ, η) pairs for which a transducer is (γ, η)-IOS;

Robustness

Several issues remain open:

- the characterization of all the (γ, η) pairs for which a transducer is (γ, η)-IOS;
- How to solve the IOS synthesis problem: existence and characterization of all the (γ, η) pairs for which there exists a controller rendering a given transducer (γ, η)-IOS;

Robustness

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- the characterization of all the (γ, η) pairs for which a transducer is (γ, η)-IOS;
- How to solve the IOS synthesis problem: existence and characterization of all the (γ, η) pairs for which there exists a controller rendering a given transducer (γ, η)-IOS;
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Robustness

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- How to make these ideas practical so that they become more useful. In particular, how to define metrics and costs in concrete problems?
- The ultimate objective is to understand robustness for cyber-physical systems.

Robustness

Relevant recent references:

- *Robust Discrete Synthesis Against Unspecified Disturbances*
  Rupak Majumdar, Elaine Render, and Paulo Tabuada
  14th International Conference on Hybrid Systems: Computation and Control 2011.
- *A theory of ω-regular robust software synthesis*
  Rupak Majumdar, Elaine Render, and Paulo Tabuada
  To appear in the ACM Transactions on Embedded Computing Systems.
- *Input-Output Robustness for Discrete Systems*
  Paulo Tabuada, Ayca Balkan, Sina Caliskan, Yasser Shoukry, and Rupak Majumdar
  International Conference on Embedded Software 2012.

For preprints and other information:

`tabuada@ee.ucla.edu`
`http://www.cyphylab.ee.ucla.edu`